Bandlimited Power of an Asynchronously Biphase-Modulated Squarewave

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Expressions for the bandlimited power of a squarewave which is biphase-modulated by an asynchronous binary data stream are determined by means of spectral integration. The utility of these expressions is demonstrated through examples using typical Mariner Venus/Mercury 1973 mission parameters.

I. Introduction

In a previous article (Ref. 1), an expression for the spectrum of a squarewave carrier biphase-modulated by an asynchronous binary data stream was determined. While this expression is useful in calculating the power spectral density at a particular frequency, it sheds little light on the calculation of the total power contained in a specified bandwidth (or equivalently, how much power is lost due to band limitations). In this article, expressions for the total power contained within an arbitrary spectral band are presented. Some sample calculations are made using Mariner/Venus Mercury 1973 mission parameters to show the utility of these expressions.

II. Derivation of Equations

Consider a squarewave carrier signal, assuming values of ± 1 with period T_S. Consider also a random binary data stream, taking on values of ± 1 with equal probability and bit period T_b. We assume that the data stream is generated asynchronously with respect to the carrier signal and that the data stream biphase-modulates the carrier. Now, if we let n denote the number of carrier half-periods which can be totally contained within one bit time, i.e.,

$$n = \text{greatest integer in} \left(\frac{2T_b}{T_S}\right) \tag{1}$$

then the spectrum of the modulated carrier $S(f)$ is given by (Ref. 1, Eq. 9)

$$S(f) = \frac{4}{T_S T_b \pi^3 f^3} \sum_{k=1}^{n} \left\{ (-1)^k \left[\sin(k\pi f T_S) - \frac{k\pi f T_S}{2} \cos(k\pi f T_S) + \pi f T_b \cos(k\pi f T_S) \right] \right\} + \frac{1}{2T_S T_b \pi^2 f^2}(T_S + 4T_b) + \frac{(-1)^n}{2\pi^2 f^2}\left[\frac{4}{T_S} - \frac{(2n+1)}{T_b}\right]\cos(2\pi f T_b) - \frac{2(-1)^n}{\pi^3 T_S T_b f^3}\sin(2\pi f T_b) \tag{2}$$

If we denote P_{f_1,f_2} as the power contained in the spectral window from f_1 to f_2 Hz, then

$$\begin{aligned} P_{f_1,f_2} &= \int_{f_1}^{f_2} S(f)\,df \\ &= \frac{4}{T_S T_b \pi^3} \sum_{k=1}^{n} (-1)^k \int_{f_1}^{f_2} \frac{\sin(k\pi f T_S)}{f^3}\,df \\ &\quad - \frac{2}{T_b \pi^2} \sum_{k=1}^{n} k(-1)^k \int_{f_1}^{f_2} \frac{\cos(k\pi f T_S)}{f^2}\,df \\ &\quad + \frac{4}{T_S \pi^2} \sum_{k=1}^{n} (-1)^k \int_{f_1}^{f_2} \frac{\cos(k\pi f T_S)}{f^2}\,df \\ &\quad + \frac{(T_S + 4T_b)}{2T_S T_b \pi^2}\left[\frac{1}{f_1} - \frac{1}{f_2}\right] \\ &\quad + \frac{(-1)^n}{2\pi^2}\left[\frac{4}{T_S} - \frac{(2n+1)}{T_b}\right] \int_{f_1}^{f_2} \frac{\cos(2\pi f T_b)}{f^2}\,df \\ &\quad - \frac{2(-1)^n}{T_S T_b \pi^3} \int_{f_1}^{f_2} \frac{\sin(2\pi f T_b)}{f^3}\,df \end{aligned} \tag{3}$$

These integrals can be evaluated by relatively straightforward techniques to yield

$$\begin{aligned} P_{f_1,f_2} &= \frac{2}{T_S T_b \pi^3} \sum_{k=1}^{n} (-1)^k \left[\frac{\sin(k\pi f_1 T_S)}{f_1^2} - \frac{\sin(k\pi f_2 T_S)}{f_2^2}\right] \\ &\quad + \frac{4}{T_S \pi^2} \sum_{k=1}^{n} (-1)^k \left[\frac{\cos(k\pi f_1 T_S)}{f_1} - \frac{\cos(k\pi f_2 T_S)}{f_2}\right] \\ &\quad + \frac{4}{\pi} \sum_{k=1}^{n} k(-1)^k \left[S_i(k\pi f_1 T_S) - S_i(k\pi f_2 T_S)\right] \\ &\quad + \frac{(-1)^n(2n+1)}{2\pi^2 T_b}\left[\frac{\cos(2\pi f_2 T_b)}{f_2} - \frac{\cos(2\pi f_1 T_b)}{f_1}\right] \\ &\quad + \frac{(-1)^n(2n+1)}{\pi}\left[S_i(2\pi f_2 T_b) - S_i(2\pi f_1 T_b)\right] \\ &\quad + \frac{(-1)^n}{T_S T_b \pi^3}\left[\frac{\sin(2\pi f_2 T_b)}{f_2^2} - \frac{\sin(2\pi f_1 T_b)}{f_1^2}\right] \\ &\quad + \frac{(T_S + 4T_b)}{2T_S T_b \pi^2}\left[\frac{1}{f_1} - \frac{1}{f_2}\right] \end{aligned} \tag{4}$$

where $S_i(x)$ is the sine integral defined by

$$S_i(x) = \int_0^x \frac{\sin u}{u}\,du \tag{5}$$

Unfortunately, no closed form has yet been determined for the sine integral. Thus, we must resort to approximate solutions. For $|x| < 1$, we can expand $S_i(x)$ in a Taylor series, so that

$$S_i(x) \approx x - \frac{x^3}{18} + \frac{x^5}{600} - \frac{x^7}{35280} + \frac{x^9}{3265920} \qquad |x| < 1 \tag{6}$$

For $|x| \geqq 1$, we can use the approximation (Ref. 2)

$$S_i(x) \approx \frac{\pi}{2} - f(x)\cos(x) - g(x)\sin(x) \qquad |x| \geqq 1 \tag{7}$$

where

$$\begin{aligned} f(x) &= \frac{1}{x}\left(\frac{x^8 + a_1x^6 + a_2x^4 + a_3x^2 + a_4}{x^8 + b_1x^6 + b_2x^4 + b_3x^2 + b_4}\right) \\ g(x) &= \frac{1}{x^2}\left(\frac{x^8 + c_1x^6 + c_2x^4 + c_3x^2 + c_4}{x^8 + d_1x^6 + d_2x^4 + d_3x^2 + d_4}\right) \end{aligned} \tag{8}$$

and the coefficients a_j, b_j, c_j, and $d_j; j = 1, 2, 3, 4$ are given in Ref. 2.

Equation (4) can be used to compute the power in any spectral band, provided neither of the frequencies f_1 or f_2 is zero. To compute the corresponding power when one of these frequencies (say, f_1) is zero, we can use the appropriate limit expression. Thus, we find that the power within the 0 to f_2 Hz band is

$$
\begin{aligned}
P_{0,f_2} &= \lim_{f_1 \to 0} \{P_{f_1,f_2}\} \\
&= -\frac{2}{T_S T_b \pi^3} \sum_{k=1}^{n} (-1)^k \frac{\sin(k\pi f_2 T_S)}{f_2^2} \\
&\quad - \frac{4}{T_S \pi^2} \sum_{k=1}^{n} (-1)^k \frac{\cos(k\pi f_2 T_S)}{f} \\
&\quad - \frac{4}{\pi} \sum_{k=1}^{n} k(-1)^k \, S_i(k\pi f_2 T_S) \\
&\quad + \frac{(-1)^n (2n+1)}{2\pi^2 T_b} \frac{\cos(2\pi f_2 T_b)}{f_2} \\
&\quad + \frac{(-1)^n (2n+1)}{\pi} S_i(2\pi f_2 T_b) \\
&\quad + \frac{(-1)^n}{T_S T_b \pi^3} \frac{\sin(2\pi f_2 T_b)}{f_2^2} - \frac{(T_S + 4T_b)}{2\pi^2 T_S T_b f_2}
\end{aligned}
\tag{9}
$$

Finally, we note that if f_2 is allowed to increase without bound, then

$$
P_{0,\infty} = \lim_{f_2 \to \infty} \{P_{0,f_2}\} = \frac{1}{2} \tag{10}
$$

which says, as one would expect, that half of the power occupies the positive frequency region.

III. Example of Band-Limited Power Loss

Consider a squarewave subcarrier of 177.1 kHz, modulated by a data stream of 117.6 kbps. This modulated signal is to be passed through a low-pass filter having a 2.0-MHz cutoff frequency. To compute the power lost due to this finite bandwidth, we can first compute the power passed by the filter from Eq. (9). If this result is doubled to account for negative frequencies, the power passed is found to be 0.959, which corresponds to a power loss of 0.18 dB.

IV. Example of Band Interference Power

Consider the same modulated subcarrier as in the previous example. This time, however, we are interested in the percentage of this signal power that occupies the region of a lower-rate channel extending from 75.483 to 102.617 kHz. Folding the spectrum and using Eq. (4) reveals that 2.1% of the high-rate signal power occupies the low-rate channel region.

References

1. Lesh, J. R., "Spectrum of an Asynchronously Biphase Modulated Squarewave," in *The Deep Space Network Progress Report,* Technical Report 32-1526, Vol. XII, pp. 226–229, Jet Propulsion Laboratory, Pasadena, Calif., Dec. 15, 1972.

2. Abramowitz, M., and Stegum, I. A., *Handbook of Mathematical Functions,* National Bureau of Standards, June 1964, pp. 232–233.